

ON TRANSFINITE NILPOTENCE OF THE VOGEL-LEVINE LOCALIZATION

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ABSTRACT. We construct a finitely-presented group such that its Vogel-Levine localization is not transfinitely nilpotent. This answers a problem of J. P. Levine.

1. INTRODUCTION

In the series of papers [8], [9], [10], [11], J. P. Levine developed the theory of algebraic closure of groups and described possible applications in geometric topology, as well as formulated natural problems related to localizations and completions of groups.

A group homomorphism $f : G \rightarrow H$ is called *2-connected* if it induces isomorphism on $H_1(-, \mathbb{Z})$ and a surjection on $H_2(-, \mathbb{Z})$. Denote by Ω the collection of all 2-connected homomorphisms $f : G \rightarrow H$ such that G and H are finitely presented groups. This concept plays a fundamental role in geometric topology, since a homology equivalence of connected spaces induces a 2-connected homomorphism of their fundamental groups (see [5], [6] for geometric applications of the theory of 2-connected homomorphisms).

A group Γ is local if given any diagram of homomorphisms as follows, with $G \rightarrow H$ from Ω ,

$$\begin{array}{ccc} G & \longrightarrow & H \\ & \searrow & \vdots \\ & & \Gamma \end{array}$$

there is a unique homomorphism $H \rightarrow \Gamma$ making the diagram commute. The *Vogel-Levine localization* (also called an *algebraic closure*) of a group G is a group $L(G)$ endowed with a homomorphism $l : G \rightarrow L(G)$, such that $L(G)$ is local and for any local group Γ and a homomorphism $f : G \rightarrow \Gamma$, there is a unique homomorphism $p : L(G) \rightarrow \Gamma$ such that $p \circ l = f$. The Vogel-Levine localization is an algebraic analog of the localization of CW-complexes considered by J.-Y. Le Dimet [7].

The Vogel-Levine localization is a functor from all groups to the local groups. The existence, uniqueness and different properties of this functor are given in [10], [8], [9]. Recall some of the properties.

- (i) Any homomorphism $f : G \rightarrow H$ from Ω induces an isomorphism of localizations $L(G) \simeq L(H)$.
- (ii) For any G , the localization $l : G \rightarrow L(G)$ is 2-connected.

This research is supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, St. Petersburg State University) under RF Government grant 11.G34.31.0026, and by JSC Gazprom Neft, as well as by the RF Presidential grant MD-381.2014.1.

For a finitely generated group G , the functor $L(G)$ lives in the corner of the following square which plays a fundamental role in the theory of localizations and completions of groups

$$(1) \quad \begin{array}{ccc} L(G) & \longrightarrow & E(G) \\ \downarrow & & \downarrow \\ \overline{G} & \longrightarrow & \widehat{G} \end{array}$$

Here $E(G)$ is the functor of HZ-localization defined by A. K. Bousfield (see [3]), $\widehat{G} := \varprojlim_i G/\gamma_i(G)$ is the functor of pro-nilpotent completion, \overline{G} is the ω -closure (or *residually nilpotent algebraic closure*, see [8], [10], [11]).

Given a finitely presented group G , it is a difficult problem how to describe the algebraic closure $L(G)$ and its group-theoretical properties. One tool to recognize $L(G)$ is given in [6]. Suppose one can construct a sequence of 2-connected homomorphisms

$$G \rightarrow K_1 \rightarrow K_2 \rightarrow \dots$$

such that K_i , $i = 1, 2, \dots$ are finitely presented and $\varinjlim K_i$ is a local group. Then $L(G) = \varinjlim K_i$. This follows immediately from the definition and uniqueness of the algebraic closure. The seed of the idea to describe the group $L(G)$ as injective limit of 2-connected maps is given in [7]. In [7], Le Dimet shows that the Vogel localization of a finite CW-complex is a colimit of a countable sequence of finite CW-complexes. The above tool to recognize $L(G)$ is an algebraic analog of the construction from [7].

For a group G , the lower central series are defined inductively as follows:

$$\gamma_1(G) = G, \quad \gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$$

and $\gamma_\tau(G) = \cap_{\alpha < \tau} \gamma_\alpha(G)$ for a limit ordinal τ . A group G is called *transfinitely nilpotent* if $\gamma_\alpha(G) = 1$ for some ordinal α . The vertical arrows in (1) are quotients of $L(G)$ and $E(G)$ by the intersections of (finite) lower central series $\gamma_\omega := \cap_i \gamma_i$.

For any group G , its HZ-localization $E(G)$ is transfinitely nilpotent [3]. In order to compare group-theoretical properties of algebraic closures and HZ-localizations, J. P. Levine asked the following ([10] (Problem 6 (b))): *If G is finitely-generated, is $L(G)$ transfinitely nilpotent?* The following example answers this problem¹.

Theorem 1. *Let $H = \langle a, b \mid a^{b^2} = aa^{3b}, [a, a^b] = 1 \rangle$. Then $L(H)$ is not transfinitely nilpotent.*

As usual, if $x, y, a_1, \dots, a_{k+1}$ are elements of a group G we set $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$ and define

$$[a_1, a_2, \dots, a_{k+1}] = [[a_1, \dots, a_k], a_{k+1}] \quad (k > 1).$$

¹In [10], J.P. Levine considered localization with respect to 2-connected maps which are normally surjective. Observe that for the group H in theorem 1, $L(G)$ is the localization in that sense as well, since all maps considered in the construction are normally surjective

2. PROOF OF THEOREM 1

The construction is based on the 1-relator group from [12]

$$G = \langle a, b \mid a^{b^2} = aa^{3b} \rangle$$

with long lower central series. Our group H is the quotient $G/\gamma_\omega(G)$. The lower central series length of G is ω^2 . The 2ω -lower central quotient

$$G/\gamma_{2\omega}(G) = \langle a, b \mid a^{b^2} = aa^{3b}, [a, a^b, a] = [a, a^b, a^b] = 1 \rangle$$

lives in the short exact sequence

$$1 \rightarrow \langle [a, a^b] \rangle \rightarrow G/\gamma_{2\omega}(G) \rightarrow H \rightarrow 1$$

where $\langle [a, a^b] \rangle$ is the infinite cyclic group, $a \in H$ acting on $[a, a^b]$ trivially and b by inverting. The lower central quotients $\gamma_{\omega+k}(G)/\gamma_{\omega+k+1}(G)$ are cyclic groups of order 2 for all $k = 0, 1, \dots$ with generators $[a, a^b]^{2^k} \cdot \gamma_{\omega+k+1}(G)$.

The proof consists of the following three steps:

- (1) Description of the ω -closure \overline{H} and the proof that $H_2(\overline{H}) = 0$. Since $\overline{H} = L(H)/\gamma_\omega(L(H))$, the part of the 5-term sequence

$$H_2(L(H)) \rightarrow H_2(\overline{H}) \rightarrow \gamma_\omega(L(H))/\gamma_{\omega+1}(L(H)) \rightarrow 1$$

implies that $\gamma_\omega(L(H)) = \gamma_{\omega+1}(L(H))$. This implies that $L(H)$ is residually nilpotent if and only if $L(H) = \overline{H}$.

- (2) A construction of the sequence of finitely presented groups Γ'_k , $k = 0, 1, 2, \dots$ and 2-connected homomorphisms

$$H = \Gamma'_0 \rightarrow \Gamma'_1 \rightarrow \dots \rightarrow \Gamma'_k \rightarrow \Gamma'_{k+1} \rightarrow \dots$$

Denote the 2-connected maps from this sequence $h_k : H \rightarrow \Gamma'_k$, $k = 1, 2, \dots$

- (3) Proof that the limit $\Gamma := \varinjlim \Gamma'_k$ is a local group. This implies that $L(H) = \Gamma$ and that the algebraic closure $H \rightarrow L(H)$ is the limit map

$$\varinjlim h_k : H \rightarrow \varinjlim \Gamma'_k.$$

It will be shown that there is a natural exact sequence

$$1 \rightarrow C_{2^\infty} \rightarrow L(H) \rightarrow \overline{H} \rightarrow 1$$

That is, $\gamma_\omega(L(H))$ is non-trivial and isomorphic to the 2-quasi-cyclic group $C_{2^\infty} := \varinjlim \{\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4 \hookrightarrow \mathbb{Z}/8 \hookrightarrow \dots\}$.

Step 1. Let $N = \mathbb{Z} \oplus \mathbb{Z}$ be the $\mathbb{Z}[\langle b \rangle]$ -module generated by a and a^b . The generator b of the cyclic quotient of H acts on N as the matrix

$$(2) \quad U := \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

Recall that (see prop. 3.2. from [11], and *telescope theorem* from [1], [2]) the ω -closure \overline{H} of H has the following natural description

$$\overline{H} = N_S \rtimes \langle b \rangle,$$

where N_S is the S -localization of N with $S := 1 + \Delta$, $\Delta = \ker\{\mathbb{Z}[\langle b \rangle] \rightarrow \mathbb{Z}\}$. By construction, N_S is the direct limit

$$N_S = \varinjlim \{N \xrightarrow{s_1} N \xrightarrow{s_2} \dots\}$$

where $\{s_1, s_2, \dots\}$ covers all elements from S .

For an element $s \in S$, the s -map $N \xrightarrow{s} N$ is injective. This follows from the residual nilpotence of H (see [12] for the proof that H is residually nilpotent). Indeed, if $s(n) = 0$, for some $n \in N$, $n \geq 0$, then $n \in \gamma_\omega(H) = 1$.

The multiplication by $s \in S$, induces a map of exterior squares

$$(\mathbb{Z} \simeq \langle a \wedge a^b \rangle \simeq) \Lambda^2(N) \rightarrow \Lambda^2(N).$$

Any homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ is a multiplication with some number. In our case denote this number by $|s|$. Let $|s| = 2^{p(s)}v$ for an odd v . Observe that $|s| \neq 0$, since, as observed above, any s -map $N \xrightarrow{s} N$ is injective.

Since $H_2(H) = \mathbb{Z}/2$, the map s induces a zero map $H_2(H) \rightarrow H_2(H)$ if and only if $p(s) \neq 0$. The simplest example of an element from S , with zero induced map $H_2(H) \xrightarrow{s} H_2(H)$ is

$$s = 1 - b + b^2.$$

Consider an element from S of the form

$$s = n_0 + n_1b + \dots + n_lb^l, \quad n_0 + \dots + n_l = 1.$$

For $a \in N$, we will use the natural notation

$$a^s := a^{n_0} a^{n_1b} \dots a^{n_lb^l}.$$

A technical exercise is to show that

$$|s| \equiv 1 + \sum_{j>i, 3|(j-i)} n_i n_j \pmod{2}$$

Since there are infinitely many indecomposed elements in S , which induce the zero map on $H_2(H)$, we conclude that

$$H_2(\overline{H}) = \varinjlim_{s_1, s_2, \dots} H_2(H) = 0.$$

Using the general relation between Vogel-Levine localization and ω -closure, $\overline{H} = L(H)/\gamma_\omega(L(H))$, the 5-term sequence implies that

$$\gamma_\omega(L(H)) = \gamma_{\omega+1}(L(H)).$$

Step 2. Denote the following groups

$$\Gamma_k := \langle a, b \mid a^{b^2} = aa^{3b}, [a, a^b, a] = [a, a^b, a^b] = 1, [a, a^b, \underbrace{b, \dots, b}_k] = 1 \rangle, \quad k \geq 1$$

Consider an element $s \in S$. For a given $k \geq 0$, we will construct a homomorphism $\phi_s : \Gamma_k \rightarrow \Gamma_{k+p(s)}$ such that there is a commutative diagram

$$\begin{array}{ccc}
 \gamma_\omega(\Gamma_k)(\simeq \mathbb{Z}/2^k) & \xrightarrow{\quad} & \gamma_\omega(\Gamma_{k+p(s)})(\simeq \mathbb{Z}/2^{k+p(s)}) \\
 \downarrow & & \downarrow \\
 \Gamma_k & \xrightarrow{\quad \phi_s \quad} & \Gamma_{k+p(s)} \\
 \downarrow & & \downarrow \\
 H & \xrightarrow{\quad s_* \quad} & H
 \end{array}$$

and ϕ_s induces an isomorphism $H_2(\Gamma_k) \rightarrow H_2(\Gamma_{k+p(s)})$.

Since the map $N \xrightarrow{s} N$ induces a well-defined homomorphism $(N \rtimes \langle b \rangle)H \rightarrow H$, there exists $l \geq 0$, such that

$$(3) \quad (a^s)^{b^2} = a^s (a^{3s})^b [a, a^b]^l$$

in Γ_k . We define the homomorphism $\phi_s : \Gamma_k \rightarrow \Gamma_{k+p(s)}$ as

$$\begin{aligned}
 a &\mapsto a^s [a, a^b]^r \\
 b &\mapsto b
 \end{aligned}$$

with r such that $3r \equiv l \pmod{2^k}$. Lets check that ϕ_s is well-defined. Indeed, the relation (3) implies that, in $\Gamma_{k+p(s)}$,

$$(a^s [a, a^b]^r)^{b^2} = (a^s [a, a^b]^r) (a^s [a, a^b]^r)^{3b}$$

The group Γ_k has another relation which we have to check. We have

$$[[a, a^b]^s, \underbrace{b, \dots, b}_k] = ([a, a^b]^s)^{2^k} = [a, a^b]^{|s|2^k} = 1$$

in $\Gamma_{k+p(s)}$. The relations $[a, a^b, a] = [a, a^b, a^b] = 1$ are preserved by the considered map. Thus, the homomorphism ϕ_s is well-defined. Observe that the homomorphism ϕ_s is normally surjective, i.e. the normal closure of the image of ϕ_s equals to $\Gamma_{k+p(s)}$.

The homology group $H_2(\Gamma_k)$ is isomorphic to $\mathbb{Z}/2$. Looking at presentation of the second homology $H_2(\Gamma_k)$ via the Hopf formula $H_2(\Gamma_k) = \frac{R \cap [F, F]}{[R, F]}$, with $F = F(a, b)$, we describe the generator of $H_2(\Gamma_k)$ as the coset

$$[a, a^b, \underbrace{b, \dots, b}_k] \cdot [R, F] = [a, a^b]^{2^k} \cdot [R, F].$$

The image of the generator of $H_2(\Gamma_k)$ under the map induced by ϕ_s is non-trivial in $H_2(\Gamma_{k+p(s)})$, hence the induced map

$$H_2(\Gamma_k)(\simeq \mathbb{Z}/2) \rightarrow H_2(\Gamma_{k+p(s)})(\simeq \mathbb{Z}/2)$$

is an isomorphism.

Lets illustrate the above construction for the particular case $s = 1 - b + b^2$. In this case, $|s| = 12$ and the induced map $H_2(H) \rightarrow H_2(H)$ is zero. We claim that

there is a commutative diagram

$$\begin{array}{ccc} & & \Gamma_2 \\ & \nearrow \phi & \downarrow \\ H & \xrightarrow{s_*} & H \end{array}$$

such that ϕ induces isomorphism $H_2(H) \rightarrow H_2(\Gamma_2)$ and the vertical map is $\Gamma_2 \rightarrow \Gamma_2/\gamma_\omega(\Gamma_2) = H$. The map ϕ is defined as

$$\begin{aligned} a &\mapsto aa^{-b}a^{b^2}[a, a^b] \\ b &\mapsto b \end{aligned}$$

One can easily see that, in Γ_2 ,

$$(aa^{-b}a^{b^2}[a, a^b])^{b^2} = aa^{-b}a^{b^2}[a, a^b](aa^{-b}a^{b^2}[a, a^b])^{3b}.$$

hence the map ϕ is well-defined.

To finalize the Step 2, we conclude that, for a sequence of elements $\{s_1, s_2, \dots\}$, there exists an infinite tower

$$\begin{array}{ccccccc} & & \mathbb{Z}/2^{|p(s_1)|} & \twoheadrightarrow & \mathbb{Z}/2^{|p(s_1)|+|p(s_2)|} & \twoheadrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \Gamma_0 & \xrightarrow{\phi_{s_1}} & \Gamma_{|p(s_1)|} & \xrightarrow{\phi_{s_2}} & \Gamma_{|p(s_1)|+|p(s_2)|} & \longrightarrow & \dots \\ \downarrow = & & \downarrow & & \downarrow & & \\ H & \xrightarrow{s_1} & H & \xrightarrow{s_2} & H & \longrightarrow & \dots \end{array}$$

All the homomorphisms ϕ_{s_i} are 2-connected. The group $\Gamma := \varinjlim_{\{s_1, s_2, \dots\}} \Gamma_{s_i}$, lies in the short exact sequence

$$1 \rightarrow C_{2^\infty} \rightarrow \Gamma \rightarrow \overline{H} \rightarrow 1$$

and the action of $\overline{H} = N_S \rtimes \langle b \rangle$ on the quasi-cyclic group C_{2^∞} is given as follows:

$$y \circ b = -y, \quad y \circ n = y, \quad n \in N_S.$$

Step 3. In order to show that Γ is local, recall the definition of local Cohn modules. For a group Q , let M be a $\mathbb{Z}[Q]$ -module. We call M a *local Cohn module* if, for every map $t : F_1 \rightarrow F_2$ of finitely generated free $\mathbb{Z}[Q]$ -modules of the same rank, such that the induced map $1 \otimes_{\mathbb{Z}[Q]} \mathbb{Z} : F_1 \otimes_{\mathbb{Z}[Q]} \mathbb{Z} \rightarrow F_2 \otimes_{\mathbb{Z}[Q]} \mathbb{Z}$ is an isomorphism, and a morphism of $\mathbb{Z}[Q]$ -modules $\alpha : F_1 \rightarrow M$, there is a unique morphism $\beta : F_2 \rightarrow M$ such that $\beta \circ t = \alpha$.

Recall the following result (see [5], [6]). Let Q be a local group and M a Cohn local $\mathbb{Z}[Q]$ -module. For any extension

$$0 \rightarrow M \rightarrow \tilde{Q} \rightarrow Q \rightarrow 1,$$

the group \tilde{Q} is local.

Observe that C_{2^∞} is the direct limit of $\mathbb{Z}[\overline{H}]$ -modules

$$\varinjlim \{\mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/8 \twoheadrightarrow \dots\}$$

Every submodule $\mathbb{Z}/2^k$, $k \geq 1$ is nilpotent $\mathbb{Z}[\overline{H}]$ -module, $(\mathbb{Z}/2^k)\Delta^k(\overline{H}) = 0$. Every nilpotent module is Cohn local and since C_{2^∞} is a direct limit of nilpotent modules, we conclude that C_{2^∞} is Cohn local module. Hence Γ is a local group and the Step 3 is complete. This completes the proof of theorem.

Acknowledgements. The author thanks S.O. Ivanov and K. Orr for discussions related to the subject of the paper.

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